Limits

Limit Laws

Given a is a constant, n is positive integer, and the limits $\lim_{x\to a} f(x), \lim_{x\to a} g(x)$ exist,

$$\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$$

$$\lim_{x\to a}[f(x) \cdot g(x)] = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$$

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad (\lim_{x\to a} g(x) \neq 0)$$

$$\lim_{x\to a} [f(x)]^n = [\lim_{x\to a} f(x)]^n$$

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$$

$$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right)$$

- Continuity at number a occurs if: $\lim_{x\to a} f(x) = f(a)$
- Types of discontinuities: jump, removable, infinite
- Logarithms grow slower than polynomials, which grow slower than exponentials.
- A limit exists if and only if its left- and right-sided limits exist and are equal.

Indeterminate forms

$$\frac{0}{0},\ \pm\frac{\infty}{\infty},\ 0\cdot\infty,\ \infty-\infty,\ 1^\infty,\ 0^0,\ \infty^0$$

Common Limits

$$\lim_{x\to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x\to 0} \frac{\cos x - 1}{x} = 0$$

$$\lim_{x\to 0} \sin\left(\frac{1}{x}\right) \quad \text{(DNE)}$$

$$\lim_{x\to 0} \frac{|x|}{x} \quad \text{(DNE)} \qquad \lim_{h\to 0} \frac{e^h - 1}{h} = 1$$

ε-δ Definition

Let $f(x)$ be a function defined on an open interval around a ($f(a)$ need not be defined).

$$\lim_{x\to a} f(x) = L$$

if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all x,

$$0 < |x-a| < \delta \implies |f(x) - L| < \epsilon.$$

Function properties

Critical value: value in domain of function f, such that $f'(c) = 0$ *or* DNE.

Concavity: based on position (above/below) of tangent line to a function in an interval.

Inflection point: point at which a function's concavity changes.

Theorems

Squeeze Theorem

$f(x) \leq g(x) \leq h(x)$ and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$

Intermediate Value Theorem

Suppose f is continuous on $[a, b]$ and let $f(a) < N < f(b)$. Then there exists c in (a, b) such that $f(c) = N$.

Increasing/Decreasing Test

If $f'(x) > 0$ / $f'(x) < 0$ on an interval, it is increasing / decreasing on that interval.

First Derivative Test

For a critical number c on continuous f, sign changes of f' determine local maximum/minimum.

Concavity Test

For a function f, if $f''(x) > 0$ / $f''(x) < 0$ for all x on an interval, f is concave up / down on that interval.

Second Derivative Test

If $f'(c) = 0$, f'' continuous near c, and $f''(c) > 0$ / $f''(c) < 0$; then f has a local minimum / maximum at c.

Differentiation

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \qquad f'(x) := \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Derivative Rules

$$\frac{dx^n}{dx} = nx^{n-1} \quad (n \in \mathbb{R})$$

$$(cf)' = cf'$$

$$\frac{df(g(x))}{dx} = \frac{df(g(x))}{dg(x)} \cdot \frac{dg(x)}{dx}$$

$$(fg)' = fg' + f'g$$

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$$

$$\log_a'(x) = \frac{1}{\ln(a)x}$$

$$(a^x)' = a^x \ln a$$

$$\ln'(x) = \frac{1}{x}$$

$$\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

$$\sin'(x) = \cos(x)$$

$$\cos'(x) = -\sin(x)$$

$$\tan'(x) = \sec^2(x)$$

$$\csc'(x) = -\csc(x)\cot(x)$$

$$\sec'(x) = \sec(x)\tan(x)$$

$$\cot'(x) = -\csc^2(x)$$

$$\arcsin'(x) = \frac{1}{\sqrt{1 - x^2}}$$

$$\arccos'(x) = -\frac{1}{\sqrt{1 - x^2}}$$

$$\arctan'(x) = \frac{1}{1 + x^2}$$

$$\operatorname{arccsc}'(x) = -\frac{1}{|x|\sqrt{x^2 - 1}}$$

$$\operatorname{arcsec}'(x) = \frac{1}{|x|\sqrt{x^2 - 1}}$$

$$\operatorname{arccot}'(x) = -\frac{1}{1 + x^2}$$

Theorems

If f is differentiable at a, then f is continuous at a.
If f is discontinuous at a, then f is not differentiable at a.

Linear Approximation

$$f'(a) \approx \frac{f(b) - f(a)}{b - a}$$ for b sufficiently close to a.

Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$,
then f has a minimum and maximum in that interval.

L'Hôpital's Rule

For functions f and g differentiable near a point a, and the limit of $f(x) / g(x)$ as x approaches a produces indeterminate form $0/0$ or ∞/∞, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the RHS limit exists or approaches $\pm\infty$.

Fermat's Theorem

If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

Rolle's Theorem

If f is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$,
then there is at least one number c in (a, b) such that $f'(c) = 0$.

Mean Value Theorem

If f is continuous on $[a, b]$, and differentiable on (a, b),
then there exists a number c in (a, b) such that $f(b) - f(a) = f'(c)(b - a)$.

Differentiation Concepts

Conjugates for limits

$$\lim_{x\to\infty}\left(\sqrt{49x^2+x}-7x\right)=\lim_{x\to\infty}\left[\left(\sqrt{49x^2+x}-7x\right)\cdot\frac{\sqrt{49x^2+x}+7x}{\sqrt{49x^2+x}+7x}\right]$$

Implicit differentiation

$$x^3+y^3=6xy \qquad\Longleftrightarrow\qquad x^3+[f(x)]^3=6xf(x)$$

$$\frac{d}{dx}\left(x^3+y^3\right)=\frac{d}{dx}x^3+\frac{dy^3}{dy}\frac{dy}{dx}=\frac{d}{dx}(6xy)$$

$$3x^2+3y^2\cdot\frac{dy}{dx}=6y+6x\frac{dy}{dx}$$

$$\frac{dy}{dx}=\frac{6y-3x^2}{3y^2-6x}$$

Related Rates

Relates the rate of change between variables based on their observed rates of interaction with other common intermediary variables. Often uses implicit differentiation and the chain rule.

Logarithmic differentiation

For differentiating functions of the form $[f(x)]^{g(x)}$.

Example: differentiate $y = x^x$.

$$\ln y = x\ln x$$

$$\frac{d}{dx}(\ln y)=\frac{d}{dx}(x\ln x)$$

$$\frac{1}{y}\frac{dy}{dx}=\ln x+1 \implies \frac{dy}{dx}=x^x(\ln x+1)$$

Examples of functions with non-differentiable points: corner, cusp, vertical tangent, and discontinuous.

One-to-one / injective function: a function that never takes the same value twice.

Inverse function: if f is an injective function with domain A and range B, then its inverse f^{-1} has domain B and range A, and is defined by $f^{-1}(y) = x$ if and only if $f(x) = y$ for any y in B.

A horizontal line test can be used to identify if a function is injective.

A vertical line test can be used to identify if the graph is of a function.

Inverse trigonometric function ranges

$$-\frac{\pi}{2}\le\sin^{-1}x\le\frac{\pi}{2} \qquad 0\le\cos^{-1}x\le\pi \qquad -\frac{\pi}{2}<\tan^{-1}x<\frac{\pi}{2}$$

Trigonometric identities

$$\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\sin\beta\cos\alpha$$
$$\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta$$

For all positive integers n,

$$a^n-b^n=(a-b)(a^{n-1}+a^{n-2}b+\cdots+ab^{n-2}+b^{n-1})$$

For all odd positive integers n,

$$a^n+b^n=(a+b)(a^{n-1}-a^{n-2}b+\cdots-ab^{n-2}+b^{n-1})$$

$$e:=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$$

Differentiation Concepts

Indeterminate Manipulation for L'Hôpital

Form: $0 \cdot \infty$

$$f(x) \cdot g(x) = \frac{f(x)}{\frac{1}{g(x)}} \quad \text{or} \quad \frac{g(x)}{\frac{1}{f(x)}}$$

Form: $\infty - \infty$

$$\lim_{x \to 1^+} \left(\frac{1}{\ln x} - \frac{1}{x-1} \right) = \lim_{x \to 1^+} \left(\frac{x - 1 - \ln x}{\ln x - x + 1} \right)$$

Indeterminate powers $0^0,\ \infty^0,\ 1^\infty$

$$y = [f(x)]^{g(x)} \implies \ln y = g(x) \ln f(x)$$

$$y = e^{\ln y}$$

Curve Sketching

1. Domain, range, intercepts, symmetry
2. Asymptotes
3. Increasing/decreasing intervals
   based on sign intervals of f'(x)
4. Local minimum/maximum
   based on sign changes of f'(x) given f''(c) ≠ 0
5. Concavity, points of inflection
   concavity based on sign of f''(x); points of inflection based on f''(x) = 0 or DNE, see if sign changes

Optimization

1. Express the problem in terms of an equation with variables representing different factors.
2. Rearrange and perform substitutions to express the maximized/minimized quantity in terms of a single variable.
3. Apply methods such as the Candidates Test (table with endpoints and critical values for closed intervals), First Derivative Test (*e.g., f'(x) > 0* for <u>all</u> *x < c,* and *f'(x) < 0* for <u>all</u> *x > c* means *f(c)* is an global maximum value of *f*), and helper tests (e.g., Second Derivative Test).

Integration Concepts

Riemann Sums

Approximates a bounded region under the interval of a function *f* as the sum of shapes. Particularly, a partition $P = \{x_0, x_1, \ldots, x_n\}$ is created by dividing an interval $[a, b]$ into n non-overlapping subintervals. This results in the general area summation (for approximating A):

$$\sum_{i=1}^{n} f(x_i^*)\Delta x_i \quad \text{where} \quad x_i^* \in [x_{i-1}, x_i];\ \Delta x_i = x_i - x_{i-1}$$

Left Riemann sum

$$L_n = \sum_{i=1}^{n} f(x_{i-1})\Delta x_i$$

Right Riemann sum

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x_i$$

Midpoint Riemann sum

$$M_n = \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x_i$$

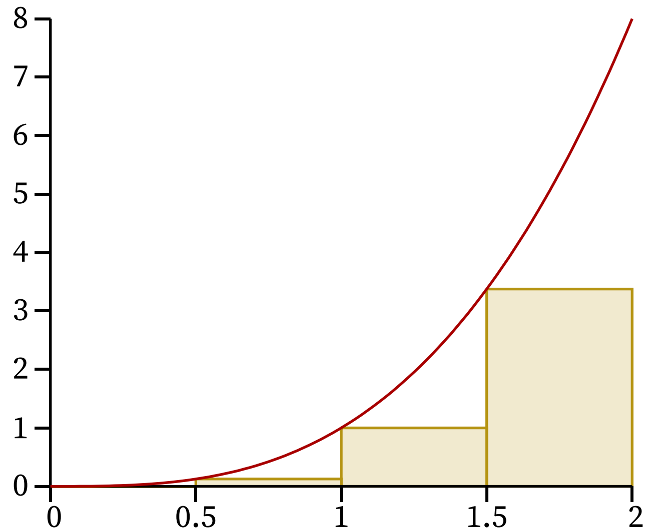
Trapezoidal sum

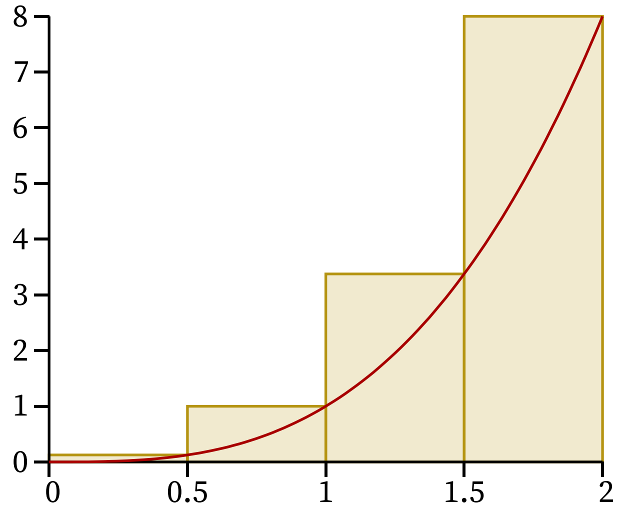
$$T_n = \sum_{i=1}^{n} \frac{f(x_{i-1})+f(x_i)}{2}\Delta x_i = \frac{L_n + R_n}{2}$$

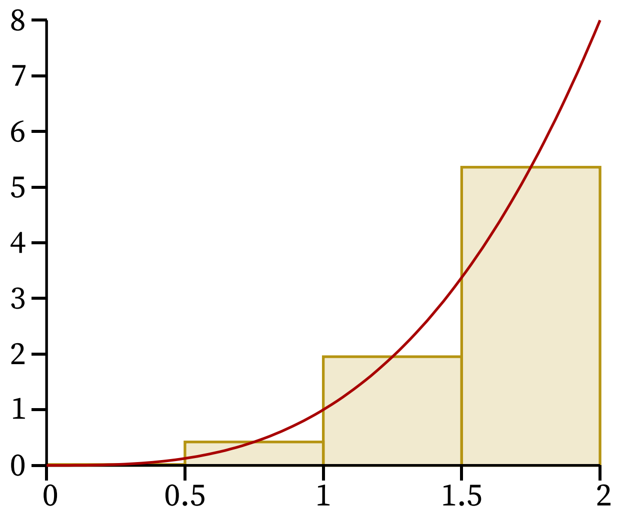
$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

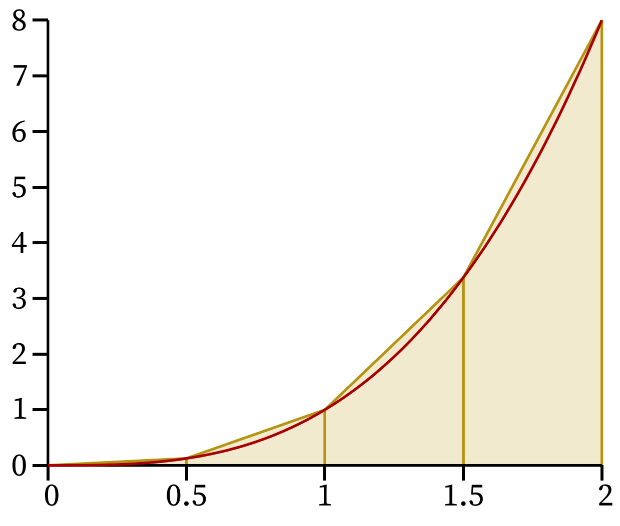
The left and right Riemann sums form upper and lower bounds for the actual area of the region. Note that the partitioning of the interval may not always be into *equal* subintervals.

Tips for ordering of different sums: consider *worst* scenario — with only one subinterval. Compare the accuracy of each sum based on the graph properties (+/–, concavity, increasing/decreasing).









Integration Concepts

Antiderivative (inverse process — finding original function given derivative)

A function F is an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x in I. The most general antiderivative if F is an antiderivative of f is $F(x) + C$ (constant).

Anti-differentiation is also called indefinite integration and is also represented as follows:

$$\int f(x)dx = F(x) + C$$

Definite integral

If f is defined on closed interval $[a, b]$, and partition $P = \{x_0, x_1, \dots, x_n\}$ is created by dividing an interval $[a, b]$ into n non-overlapping subintervals, and $x_i^* \in [x_{i-1}, x_i]$; $\Delta x_i = x_i - x_{i-1}$

Then, the definite integral of f from a to b is

$$\int_a^b f(x)dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x_i$$

provided the limit exists, and gives the same value for all possible choices of sample points. If the limit does exist, then f is *integrable* on $[a, b]$. $f(x)$ is denoted the *integrand*; a, b are called the *lower* and *upper limits of integration*, respectively.

Note that the definite integral calculates the net area — the difference of the areas of the graph above the x-axis (and below f), and below the x-axis (and above f).

Conditions for integrability

If f is continuous on $[a, b]$ or f has a finite number of jump discontinuities, then f is integrable on $[a, b]$.

Function	*Antiderivative*
$cf(x)$	$cF(x)$
$f(x) + g(x)$	$F(x) + G(x)$
x^n $(n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln\lvert x\rvert$
e^x	e^x
b^x	$\frac{b^x}{\ln b}$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\sec^2 x$	$\tan x$
$\sec x \tan x$	$\sec x$
$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$
$\frac{1}{1+x^2}$	$\tan^{-1} x$

Summation Formulas

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left[\frac{n(n+1)}{2}\right]^2$$

$$\sum_{i=1}^{n} i^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$$

Applications

Area, distance, rectilinear motion, gravity/velocity, volume

Definite Integral Properties

$$\int_a^b c\, dx = c(b-a)$$

$$\int_a^b [f(x) \pm g(x)]\, dx = \int_a^b f(x)\, dx \pm \int_a^b g(x)\, dx$$

$$\int_a^b cf(x)\, dx = c\int_a^b f(x)\, dx$$

$$\int_a^b f(x)\, dx = \int_a^c f(x)\, dx + \int_c^b f(x)\, dx$$

If $f(x) \geq 0$ for $a \leq x \leq b$, $\int_a^b f(x)\, dx \geq 0$

If $f(x) \geq g(x)$ for $a \leq x \leq b$, $\int_a^b f(x)\, dx \geq \int_a^b g(x)\, dx$

If $m \leq f(x) \leq M$ for $a \leq x \leq b$,

$$m(b-a) \leq \int_a^b f(x)\, dx \leq M(b-a)$$

Integration

Fundamental Theorem of Calculus

If f is a continuous function on $[a, b]$, then:

1. If $g(x) = \int_a^x f(t)\, dt$ then $g'(x) = f(x)$
2. $\int_a^b f(x)\, dx = F(b) - F(a)$, where $F' = f$.

Symmetry

For f that is continuous on $[-a, a]$,

If $f(-x) = f(x)$, then $\int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx$

If $f(-x) = -f(x)$, then $\int_{-a}^{a} f(x)\, dx = 0$

Table of Indefinite Integrals (constant of integration C is omitted)

$$\int cf(x)\, dx = c\int f(x)\, dx \qquad \int [f(x) + g(x)]\, dx = \int f(x)\, dx + \int g(x)\, dx$$

$$\int k\, dx = kx \qquad \int x^n\, dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$$

$$\int \frac{1}{x}\, dx = \ln|x| \qquad \int e^x\, dx = e^x \qquad \int b^x\, dx = \frac{b^x}{\ln b}$$

$$\int \sin x\, dx = -\cos x \qquad \int \cos x\, dx = \sin x$$

$$\int \sec^2 x\, dx = \tan x \qquad \int \csc^2 x\, dx = -\cot x$$

$$\int \sec x \tan x\, dx = \sec x \qquad \int \csc x \cot x\, dx = -\csc x$$

$$\int \sec x\, dx = \ln|\sec x + \tan x| \qquad \int \csc x\, dx = \ln|\csc x - \cot x|$$

$$\int \tan x\, dx = \ln|\sec x| \qquad \int \cot x\, dx = \ln|\sin x|$$

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) \qquad \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right), \text{ for } a > 0$$

$$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| \qquad \int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right|$$

'Chain Rule' – Example: for $m(x) < 0 < n(x)$,

$$\frac{d}{dx}\left[\int_{m(x)}^{n(x)} f(t)\, dt\right] = -\frac{d}{dt}\left[\int_0^{m(x)} f(t)\, dt\right] + \frac{d}{dt}\left[\int_0^{n(x)} f(t)\, dt\right] = -f(m(x)) \cdot \frac{dm}{dx} + f(n(x)) \cdot \frac{dn}{dx}$$

Applications of Integration

Area between curves

The **area** A of the region bounded by the graphs of $y = f(x)$ and $y = g(x)$ and the lines $x = a$ and $x = b$, where f and g are continuous and $f(x) \geq g(x)$ for all x in $[a, b]$, is

$$A = \int_a^b [f(x) - g(x)]\,dx$$

When using a calculator, can consider $|f(x) - g(x)|$ if $f(x)$ is not always larger than $g(x)$.

Sometimes, you may be asked to find area for functions expressed in the form $x = f(y)$. Consider its graph and decide whether to integrate with respect to x or y.

Volume

Let S be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of S in the plane P through x that is perpendicular to the x-axis is $A(x)$ (where A is continuous), then the volume of S is

$$V = \lim_{n\to\infty} \sum_{i=1}^{n} A(x_i^*)\Delta x = \int_a^b A(x)\,dx$$

Consider using cross-sectional shapes such as circles, semicircles, equilateral triangles, squares, and rectangles.

Disk, washer methods

For a solid of revolution (where solid can be formed by revolving an area around an axis of rotation), volume can be calculated as sum of cylindrical cross-sections. If the solid is hollowed by another solid of revolution, simply subtract its volume from that of the larger solid.

Substitution Rule

If $u = g(x)$ is a differentiable function with range I and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Average value of a function

The average value of f on the interval $[a, b]$ is

$$f_{\text{avg}} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

Mean value theorem for integrals

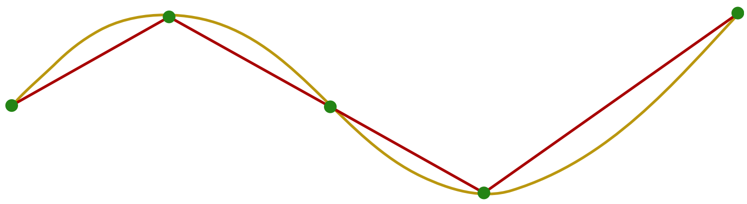
If f is a continuous function on $[a, b]$, then there exists a number c in $[a, b]$ such that

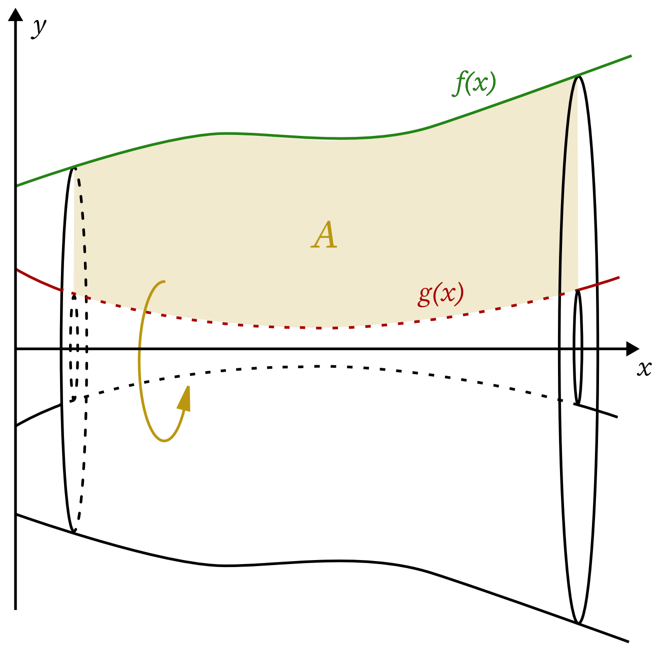
$$\int_a^b f(x)\,dx = f(c)(b-a)$$

Arc length

If f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$, $a \leq x \leq b$, is

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$





Substitution Rule Tips

1. Choose a substitution $u = g(x)$.
2. Compute $du = g'(x)\,dx$.
3. Rewrite the integral in terms of u and du.
4. Evaluate the integral in terms of u.
5. Replace u by $g(x)$ to obtain an antiderivative in terms of x.
6. Check your solution by differentiating (remember C!).

Techniques of Integration

Trigonometric integrals

$$\int \sin^m x \, \cos^n x \; dx$$

(a) n is odd: save one cosine factor, use $cos^2x = 1 - sin^2x$ to express remaining factors as sine, use $u = sin\ x$ substitution.

(b) m is odd: save one sine factor, use $sin^2x = 1 - cos^2x$ to express remaining factors as cosine, use $u = cos\ x$ substitution.

(c) n and m both even: use half-angle identities.
$sin^2x = 0.5(1 - cos\ 2x)$
$cos^2x = 0.5(1 + cos\ 2x)$
$sin\ x\ cos\ x = 0.5\ sin\ 2x$

$$\int \tan^m x \, \sec^n x \; dx$$

(a) n is even: save sec^2x factor, express remaining as $sec^2x = 1 + tan^2x$, use $u = tan\ x$ substitution.

(b) m is odd: save $sec\ x\ tan\ x$ factor, express remaining tan in terms of sec with $tan^2x = sec^2x - 1$, use $u = sec\ x$ substitution.

$$\int \sin mx \cos nx \; dx \qquad \int \sin mx \sin nx \; dx$$

$$\int \cos mx \cos nx \; dx$$

Evaluate with corresponding identity:
(a) $sin\ A\ cos\ B = 0.5[sin(A - B) + sin(A + B)]$
(b) $sin\ A\ sin\ B = 0.5[cos(A - B) - cos(A + B)]$
(c) $cos\ A\ cos\ B = 0.5[cos(A - B) + cos(A + B)]$

Trigonometric substitution

Expression	Substitution
$\sqrt{a^2 - x^2}$	$x = a\sin\theta, \text{ for } -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta, \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2}$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta,$ for $0 \leq \theta < \frac{\pi}{2}$ or $\pi \leq \theta < \frac{3\pi}{2}$

Integration by parts

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$$

LIATE
Logarithm
Inverse trigonometric functions
Algebraic functions (polynomials)
Trigonometric functions
Exponential functions
Let u = term appearing highest in LIATE.

Variations of integration by parts:
1. Multiply by 1

$$\int \ln x \; dx \to u = \ln x, \; dv = 1$$

2. Loop round

$$\begin{aligned} I &= \int e^x \sin x \; dx \\ &= e^x \sin x - \int e^x \cos x \; dx \\ &= e^x \sin x - \left(e^x \cos x + \int e^x \sin x \; dx\right) \end{aligned}$$

$$\therefore 2I = e^x(\sin x - \cos x)$$

3. By parts twice

$$\int x^2 \sin x \; dx$$

$$= -x^2 \cos x + \int 2x \cos x \; dx$$

$$= -x^2 \cos x + 2x \sin x - \int 2 \sin x \; dx$$

Partial fraction decomposition

Denominator term	Term in PFD
$(ax+b)^k$	$\frac{A_1}{ax+b} + \ldots + \frac{A_k}{(ax+b)^k}$
$(ax^2+bx+c)^k$ *(irreducible quadratic term)*	$\frac{A_1x+B_1}{ax^2+bx+c} + \ldots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$

Improper Integrals

Improper integrals

If $\int_a^t f(x)\,dx$ exists for every number $t \geq a$, then (provided limit exists)

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx, \text{ and similarly, } \int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$$

The improper integral is termed *convergent* if the limit corresponding exists, and *divergent* if it does not. Improper integrals with limits at $\pm\infty$ can be evaluated (for any real number a) as

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

If a function $f(x)$ has a discontinuity at c, where $a < c < b$, then for converging resultant integrals,

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \\ &= \lim_{t\to c^-}\int_a^t f(x)\,dx + \lim_{t\to c^+}\int_t^b f(x)\,dx\end{aligned}$$

Comparison theorem

For continuous functions f and g with $f(x) \geq g(x) \geq 0$ *for* $x \geq a$, then

$$\int_a^\infty f(x)\,dx \text{ converges} \implies \int_a^\infty g(x)\,dx \text{ converges}$$

$$\int_a^\infty g(x)\,dx \text{ diverges} \implies \int_a^\infty f(x)\,dx \text{ diverges}$$

Power Series $\sum_{n=0}^\infty c_n(x-a)^n$

A power series centered at a has only three possibilities for convergence:
(a) The series converges only when $x = a$.
(b) The series converges for all x.
(c) There is a positive R (*radius of convergence*) such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

If a power series has $R > 0$, then it is differentiable (and therefore continuous) on $(a - R,\ a + R)$, and the radius of convergence of the power series formed by its derivative and indefinite integral are both also R.

Taylor series

$$f(x) = \sum_{n=0}^\infty \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Taylor inequality

If $|f^{(n+1)}(x)| \leq M$ for $|x - a| \leq d$, then (for $|x - a| \leq d$)

$$|f(x) - T_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

Differential Equations

First-order differential equations

1. Separable

$$\frac{dy}{dx} = f(x)g(y) \implies \int \frac{dy}{g(y)} = \int f(x)\,dx$$

2. Homogenous

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right), \text{ let } y = vx \implies \frac{dy}{dx} = x\frac{dv}{dx} + v$$

3. Integrating factor

$$\frac{dy}{dx} + P(x)y = Q(x), \text{ let } I = e^{\int P(x)\,dx}$$

$$\int I\left(\frac{dy}{dx} + P(x)y\right)dx = Iy = \int IQ(x)\,dx$$

Euler's method

Approximate values for the solution to a differential equation $y' = F(x, y)$, given that $y(x_0) = y_0$ with step size h, at $x_n = x_{n-1} + h$, are $y_n = y_{n-1} + h \cdot F(x_{n-1}, y_{n-1})$.

Infinite Sequences and Series

Notation

$\{a_1, a_2, \ldots\}, \{a_n\}, \text{ or } \{a_n\}_{n=1}^{\infty}$

Monotonic sequence theorem

Every bounded, monotonic sequence is convergent.

Squeeze theorem for sequences

If for all $n \geq n_0, a_n \leq b_n \leq c_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L \implies \lim_{n\to\infty} b_n = L$

Terminology

A sequence *converges* if as $n \to \infty$, the nth term approaches a fixed value, otherwise it *diverges*.
A sequence is *bounded* if all terms are within a fixed range (*bounded above* and *below*), else *unbounded*.
A sequence is *increasing* if all successive terms are larger than previous terms, similarly for *decreasing*.
A sequence is *monotonic* if and only if it is an increasing or decreasing sequence.
A series is *alternating* if its terms are alternating between positive and negative.
A series is *absolutely convergent* if the series with the absolute values of its terms is convergent.
A series is *conditionally convergent* if it converges but is not absolutely convergent.

Test for divergence

If $\lim_{n\to\infty} a_n$ does not exist or $\neq 0$, then $\sum_{n=1}^{\infty} a_n$ is a divergent series.

Integral test

For $N \in \mathbb{Z}$ and a function f defined on $[N, \infty)$, on which it is monotonically decreasing,
then $\sum_{n=N}^{\infty} f(n)$ converges to a real number if and only if $\int_N^{\infty} f(x)\,dx = F, F \in \mathbb{R}$.

Comparison test

Suppose Σa_n and Σb_n are series with positive terms.
(a) If Σb_n converges and $a_n \leq b_n$ for all n, then Σa_n converges.
(b) If Σb_n diverges and $a_n \geq b_n$ for all n, then Σa_n diverges.

p-series test

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$.

Limit comparison test

Suppose Σa_n and Σb_n are series with positive terms.
For a finite number $c > 0$, if

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

then either both series converge or both diverge.

Ratio test

If the value of

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$$

(a) is < 1, then the series is absolutely convergent and thus also convergent.
(b) is > 1, the series is divergent.
(c) is $= 1$, then the test is inconclusive.

Alternating series test

An alternating series converges if the magnitude of successive terms decreases and approaches 0.

Root test

If the value of

$$\lim_{n\to\infty} \sqrt[n]{|a_n|}$$

(a) is < 1, then the series is absolutely convergent and thus also convergent.
(b) is > 1, the series is divergent.
(c) is $= 1$, then the test is inconclusive.

Parametric Equations and Polar Coordinates

Parametric equation notation

A *parametric curve C* can be described by *parametric equations* relating points (x, y) as functions of parameter t such as $x = f(t)$ and $y = g(t)$, along some interval $a \leq t \leq b$, starting at *initial point* $(f(a), g(a))$ and ending at *terminal point* $(f(b), g(b))$.

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$$

Arc length

If a curve C is described by $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Area

If a curve C is described by $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$, where C is traversed exactly once as t increases from α to β, then the area under the curve is

$$A = \int_{\alpha}^{\beta} g(t) f'(t)\, dt$$

Polar coordinates

The point (r, θ) describes a point r away from the origin, at an angle θ rotated counter-clockwise from the x-axis. Notice that $x = r \cos \theta$ and $y = r \sin \theta$, and $r^2 = x^2 + y^2$ and $\tan \theta = y/x$.

$$\frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Polar curves

The graph of a polar equation $r = f(\theta)$ consists of all points P whose coordinates (r, θ) satisfy the equation.

Arc length

The length of a polar curve described by $r = f(\theta)$, for $a \leq \theta \leq b$ is

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Area

The graph of a polar equation $r = f(\theta)$ consists of all points P whose coordinates (r, θ) satisfy the equation.

$$A = \int_{a}^{b} \frac{1}{2} r^2\, d\theta$$